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ON THE THEORY OF DYNAMIC THERMOELASTICITY

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Use of the causality principle as radiation condition in dynamical problems of thermoelasticity is proposed. It follows from an analysis of the fundamental mathematical models describing the thermoelastic behavior of a continuous medium and used in the solution of specific problems, that some will yield physically unrealizable solutions. To eliminate the ambiguity in the solution which occurs, an approach is possible which has an explicit physical meaning and is based on the causality principle [1, 2]; it is required that the time source not yield a response earlier than the time of starting up of the source. Different kinds of radiation conditions of the Sommerfeld type are known in thermoelasticity problems [3 — 6].

To extract the unique solution in dynamical thermoelasticity problems, it is proposed in this paper to use the causality principle, which is equivalent to the requirement of analyticity of the solution in the upper half of the complex frequency plane; there are studied the analytic properties of the solutions of the fundamental boundary value problems for the models used most often for thermoelastic media, and there are made deductions about their physical realizability.

The fundamental equations of the generalized dynamical coupled problem in the absence of mass forces and heat sources have the form [7]

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \beta\,\mathrm{grad}\,t - \rho\frac{\partial^2\mathbf{u}}{\partial\tau^2} = 0 \tag{1}$$

$$\Delta t - \frac{1}{a}\frac{\partial t}{\partial\tau} - \frac{1}{c_q^2}\frac{\partial^2 t}{\partial\tau^2} - \eta\,\mathrm{div}\frac{\partial\mathbf{u}}{\partial\tau} - \eta\tau_r\,\mathrm{div}\frac{\partial^2\mathbf{u}}{\partial\tau^2} = 0$$

The notation from the paper [7] is used here.

If the expressions for the displacement and temperature are represented as

$$\mathbf{u}(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{U}_F(x,\omega)\,e^{i\omega\tau}d\omega$$

$$t(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}T_F(x,\omega)\,e^{i\omega\tau}d\omega$$

then $\mathbf{U}_F(x,\omega)$ and $T_F(x,\omega)$ will satisfy the following system of equations:

$$\mu\Delta \mathbf{U}_F + (\lambda + \mu)\,\mathrm{grad\,div}\,\mathbf{U}_F - \beta\,\mathrm{grad}\,T_F + \rho\omega^2\,\mathbf{U}_F = 0 \qquad (2)$$
$$\Delta T_F + \left(\frac{i\omega}{a} + \frac{\omega^2}{c_q^2}\right)T_F + \eta(1 - i\omega\tau_r)\,i\omega\,\mathrm{div}\,\mathbf{U}_F = 0$$

Exactly as in [3], it can be shown for the system (2) that its regular solution allows the following representation in the domain of regularity:

$$\mathbf{U}_F = \mathbf{U}_1 + \mathbf{U}_2$$
$$(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)\mathbf{U}_1 = 0, \quad \mathrm{rot}\,\mathbf{U}_1 = 0$$
$$(\Delta + \lambda_3^2)\mathbf{U}_2 = 0, \quad \mathrm{div}\,\mathbf{U}_2 = 0$$
$$(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)T_F = 0$$
$$\lambda_1^2 + \lambda_2^2 = \frac{\omega^2}{c_1^2} + i\omega\left(\frac{1}{a} - \frac{i\omega}{c_q^2}\right) + \frac{i\omega(1 - i\omega\tau_r)}{\lambda + 2\mu}\eta\beta$$
$$\lambda_1^2\lambda_2^2 = i\omega\frac{\omega^2}{c_1^2}\left(\frac{1}{a} - \frac{i\omega}{c_q^2}\right), \quad \lambda_3^2 = \frac{\omega^2}{c_2^2}$$

Investigation shows that $\lambda_k(s)$ ($k = 1, 2, 3$; $-s = i\omega$) are analytic functions of the parameter s for $\mathrm{Re}\,s > 0$ if the following condition is satisfied:

$$c_1^2 \geqslant \frac{1 - \varepsilon}{(1 + \varepsilon)^2}c_q^2, \quad \varepsilon = \frac{a\eta\beta}{\lambda + 2\mu} \qquad (3)$$

If σ is the index of exponential growth in time of the initial data in the problem and its required derivatives equals zero and the inequality (3) is satisfied, then (see [8], Ch.), the unique solution of the boundary value problem will be (S is a piecewise-smooth surface bounding a finite domain)

$$\mu\Delta \mathbf{U}_L + (\lambda + \mu)\,\mathrm{grad\,div}\,\mathbf{U}_L - \beta\,\mathrm{grad}\,T_L - \rho s^2\mathbf{U}_L = \mathbf{f}_1(x, s) \qquad (4)$$
$$\Delta T_L - \left(\frac{s}{a} + \frac{s^2}{c_q^2}\right)T_L - \eta(1 + s\tau_r)\,s\,\mathrm{div}\,\mathbf{U}_L = f_2(x, s)$$
$$\mathbf{U}_L\Big|_S = \mathbf{F}_1(x, s), \quad T_L\Big|_S = F_2(x, s)$$
$$\mathbf{U}_L(x, s) = \int_0^\infty \mathbf{u}(x, \tau)e^{-s\tau}d\tau, \quad T_L(x, s) = \int_0^\infty t(x, \tau)e^{-s\tau}d\tau$$

Here $\mathbf{U}_L(x, s)$, $T_L(x, s)$ are analytic functions for $\mathrm{Re}\,s > 0$. The assertion that $\mathbf{U}_L(x, s)$ is a regular function for $\mathrm{Re}\,s > 0$ is equivalent to the assertion that the appropriate Fourier transform $\mathbf{U}_F(x, \omega)$ is regular for $\mathrm{Im}\,\omega > 0$. The uniqueness theorem for the generalized dynamical coupled problem of thermoelasticity is proved in [9].

When $\varepsilon = 0$, separation of the deformation and temperature fields occurs. Then compliance with the inequality

$$c_1 \geqslant c_q \qquad (5)$$

is necessary for analyticity of the appropriate solution of the generalized uncoupled problem in the upper half-plane of the complex frequency plane.

In the classical case, the problem (1) goes over into the dynamical coupled problem. The analytical properties of the unique solution of the boundary value problem obtained from (4) for $c_q = \infty$, $\tau_r = 0$ are studied in [8]. It is shown that if the index of exponential growth in time of the initial data of the problem is zero, then the solution is analytic for $\operatorname{Re} s > \sigma_0$ or, equivalently, U_F and T_F are analytic for $\operatorname{Im} \omega > \sigma_0$, where

$$\sigma_0 = \max \{c_1^2 a^{-1}(1 - \varepsilon),\ 0\} \tag{6}$$

Hence, a necessary criterion for the analyticity of the causality principle is satisfied just under the condition $\varepsilon \geqslant 1$. Thus, for $\varepsilon = 0$ when separation of the deformation and temperature fields occurs, the solution of the dynamical uncoupled problem will already not be an analytic function for $\operatorname{Im} \omega > 0$.

Neglecting the inertial terms in (1) results in a generalized quasistatic coupled problems, which we write in the space of Fourier time transforms

$$\begin{gathered} \mu \Delta \mathbf{U}_F + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{U}_F - \beta \operatorname{grad} T_F = 0 \\ \Delta T_F + \left(\frac{i\omega}{a} + \frac{\omega^2}{c_q^2}\right) T_F + \eta (1 - i\omega\tau_r) i\omega \operatorname{div} \mathbf{U}_F = 0 \end{gathered} \tag{7}$$

We represent the solution of (7) in the form

$$\mathbf{U}_F = \mathbf{U}_1 + \mathbf{U}_2 + \mathbf{U}_3$$

Here $\mathbf{U}_1$ is the solution of the temperature-free static force problem with given boundary conditions, which evidently satisfies the causality principle; $U_2 = \operatorname{grad} \Phi_F$ is a particular solution of the first equation in (7), and $\mathbf{U}_3$ is a solution introduced in order to satisfy the zero boundary conditions in combination with $\mathbf{U}_2$. For Φ_F we obtain the equation

$$\Delta \Phi_F = k T_F, \quad k = \frac{\beta}{\lambda + 2\mu}$$

The equation to determine T_F then follows from the second equation in (6):

$$\begin{gathered} \Delta T_F + (i\omega M + \omega^2 N) T_F = 0 \\ M = a^{-1} + \eta k, \quad N = c_q^{-2} + \tau_r \eta k \end{gathered} \tag{8}$$

where

$$\Phi_F = -\frac{k}{i\omega M + \omega^2 N} T_F \tag{9}$$

It is known that the solution of the Helmholtz equation which T_F satisfies for the fundamental boundary value problems is an analytic function of the parameter ω for $\operatorname{Im} \omega > 0$. It then follows from (9) that Φ_F is also an analytic function in the upper half-place of the complex frequency plane. The operations which must be performed with the function Φ_F in order to obtain the stresses and displacements also result in functions with the mentioned analyticity property. The displacements or stresses, expressed in terms of the function U_3 , cancel the stresses or displacements expressed in terms of the potential Φ_F on the boundary of the domain S . Therefore, the boundary conditions for $\mathbf{u}_3$ are zero conditions for $\tau < 0$. Use of the uniqueness theorem results in the deduction that $u_3 \equiv 0$ for $\tau < 0$. Then $u \equiv 0$ up to the beginning of the effect of the perturbation.

This deduction remains valid even for particular cases of the problem (7): $\eta = 0$ is the generalized quasistatic uncoupled problem; $c_q = \infty$, $\tau_r = 0$ is the quasistatic coupled problem; $c_q = \infty$, $\tau_r = 0$, $\eta = 0$ is the quasistatic uncoupled problem.

In order to clarify the subsequent discussion better and to relate them to the facts already known, let us examine a known example from physics which is considered in detail in [10]. The equation of motion for the displacement x_j of the j-th electron subjected to the incident wave field $E = E_\omega e^{-i\omega\tau}$ has the form

$$m\left(\frac{d^2 x_j}{d\tau^2} + 2\gamma_j \frac{dx_j}{d\tau} + \omega_j^2 x_j\right) = eE_\omega e^{-i\omega\tau}, \quad \gamma_j > 0 \tag{10}$$

where the second term in parentheses characterizes the damping due to collisions and radiation, and ω_j is the frequency of the natural vibrations. The analytic solution of (10) for $\operatorname{Im}\omega > 0$ has the form

$$x_j = \frac{eE_\omega e^{-i\omega\tau}}{m\left(\omega_j^2 - 2i\gamma_j\omega - \omega^2\right)} \tag{11}$$

If the decelerating effect exerted by the electron is due just to radiation, then according to Lorentz, the term characterizing the damping in (10) can be replaced by the radiation reaction $-\beta_j d^3x_j / d\tau^3$. This results in replacement of the term $-2i\gamma_j\omega$ by $-i\beta_j\omega^3$. The expression obtained already does not satisfy the causality principle since there is a pole $\omega \approx i / \beta_j$ $(1/\beta_j \gg \omega_j)$ in the upper half-plane $\operatorname{Im}\omega > 0$. Taking this into account, the deduction is made that the Lorentz refinement is not justified.

Similar difficulties are encountered in the dynamical problems of thermoelasticity. Functions satisfying the causality principle are the solution of the Lamé equation without a temperature term, as follows from the analytic properties of the solution of the corresponding elliptical problem [8]. The solution of boundary value problems for the diffusion form of the heat conduction equation (resulting from the Fourier law), which yields a sufficiently accurate description of the temperature field except for short time intervals, also does not violate the causality principle. Since the temperature will have an infinite propagation velocity, then the temperature gradient, and hence a quantity proportional to the gradient $(-\beta \operatorname{grad} t)$ will have an infinite propagation velocity. An attempt to refine the Lamé equations taking account of the finite propagation velocity for the compression and shear waves by appending a parabolic heat conduction equation and adding the term $-\beta \operatorname{grad} t$ to the Lamé equations (the system (1) for $c_q = \infty$, $\tau_r = 0$, $\eta = 0$) results in the already mentioned unacceptable property: the solution of this problem will not satisfy the causality principle.

Certain improvements are observed upon insertion of a term taking account of the mechanical coupling in the heat conduction equation

$$-\frac{\beta t_0}{\lambda_t}\frac{\partial}{\partial\tau}\operatorname{div}\mathbf{u} \tag{12}$$

(dynamical coupled problem). All these improvements are insufficient for compliance with the causality principle since the value of ε is within the interval $(0, 1)$ and is considerably less than one for the majority of real bodies [11, 12]. Hence, the values of ε do not result in the equality $\sigma_0 = 0$ (see (6)), i.e., the analyticity of the solution in the upper half-plane of the complex variable ω.

If a heat conduction equation with an infinite propagation velocity for the heat is added to the static Lamé equation with the term $-\beta \operatorname{grad} t$ which can be interpreted as a dynamical Lamé equation with infinite propagation velocity for the compression and shear waves, with or without the mechanical coupling term taken into account, then the quasistatic problem obtained will describe the behavior of a thermoelastic medium without violating the causality principle.

The deformation field in the body can be represented as the sum of a potential and vortex field. Then only the potential field component will be related to the temperature. Taking account of the preceding reasoning, it can be foreseen that the mathematical model of a medium taking account of the finite shear wave velocity and the infinite compression wave and temperature propagation velocity

$$\Delta\varphi - kt = 0, \quad \Delta\boldsymbol{\psi} - \frac{1}{c_2^2}\frac{\partial\boldsymbol{\psi}}{\partial\tau^2} = 0, \quad \operatorname{div}\boldsymbol{\psi} = 0 \tag{13}$$

$$\Delta t - M\frac{\partial t}{\partial\tau} = 0, \quad \mathbf{u} = \operatorname{grad}\varphi + \operatorname{rot}\boldsymbol{\psi}$$

will also not violate the causality principle. Indeed, reasoning exactly as in the case of the quasistatic problem, we obtain that the solution of the boundary value problem for (13) in the Fourier time-transform space will be an analytic function of the parameter ω for $\operatorname{Im}\omega > 0$.

We call the problem (13) a quasidynamical coupled problem. The mentioned analyticity property is evidently conserved even for the quasidynamical uncoupled problem when the mechanical coupling term in the heat conduction equation is not taken into account, i.e., for $M = 1/a$.

If it is taken into account that the heat flux depends not only on the temperature gradient but also on its prehistory, then we obtain a modified Fourier law

$$\mathbf{q} + \tau_r\frac{\partial}{\partial\tau}\mathbf{q} = -\lambda_t \operatorname{grad} t$$

which Maxwell actually described in 1867 [13]. This law results in a hyperbolic heat conduction equations, which corresponds to a finite heat propagation velocity equal to $c_q = \sqrt{a/\tau_r}$. The condition for compliance with the causality principle for the dynamical coupled problem of thermoelasticity is given by (3) when a finite heat propagation velocity (second sound) is taken into account. But it was noted in [7] that the experimental data on thermal pulsations indicate that the acoustic wave front leads the thermal front, i.e., a stronger inequality holds for a real body than the necessary (3) or (5), $c_1 > c_q$.

We then obtain that the causality principle is satisfied for both the generalized dynamical coupled, and the uncoupled, problems.

Summarizing, we conclude that the causality principle is satisfied for the following problems: the generalized dynamical coupled and uncoupled problems, the generalized quasidynamical coupled and uncoupled problems, the generalized quasistatic coupled and uncoupled problems, the quasistatic coupled and uncoupled problems, but is not satisfied for the dynamical coupled and dynamical uncoupled problems.

Therefore, if the magnitude of the heat propagation velocity in the thermoelasticity equations is greater than its corresponding velocity of compression wave propagation, the solution of the corresponding boundary value problem, except the generalized dynamical coupled problem, will cease to portray the behavior of a real thermoelastic body even within the framework of those inaccuracies which follow directly from the physical assumptions. This problem can be used to describe the thermoelastic behavior of media for which the heat propagation velocity insignificantly exceeds the velocity of compression wave propagation (if such media exist).

Uncertainties in the selection of the branch cut direction and the bypass direction of the singularities on the real axis will occur in solving specific dynamical thermoelasticity problems by using Fourier integral transforms in the time and Hankel or Fourier transforms in the space coordinates, exactly as in the case of the dynamical elasticity theory problems [2]. Utilization of the causality principle to this end is in no way different from its application in the dynamical elasticity theory problems considered in [2].

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